A Characterization of Strong Approximation Resistance

Algorithm

Hardness

zero-sum game

value = 0

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Joint work with
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Max-k-CSP

- "Satisfy as many as possible."
Max-$k$-CSP

- $n$ Boolean variables, $m$ constraints (each on $k$ variables)
Max-k-CSP

- $n$ Boolean variables, $m$ constraints (each on $k$ variables)
- Satisfy as many as possible.

Max-3-SAT

\begin{align*}
    x_1 & \lor x_2 \lor \overline{x}_{19} \\
    x_3 & \lor \overline{x}_9 \lor x_{23} \\
    x_5 & \lor \overline{x}_7 \lor \overline{x}_9 \\
    \vdots & \quad \vdots
\end{align*}
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Max-3-SAT

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\vdots & \\
\vdots &
\end{align*}
\]

Max-Cut

\[
\begin{align*}
\text{Graph with nodes } & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \\
\text{edges } & (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_3), (x_2, x_4), (x_2, x_5), (x_3, x_4), (x_3, x_5), (x_4, x_5)
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Max-\( k \)-CSP

- \( n \) Boolean variables, \( m \) constraints (each on \( k \) variables)
- Satisfy as many as possible.

**Max-3-SAT**

\[
x_1 \lor x_{22} \lor \overline{x}_{19} \\
x_3 \lor \overline{x}_9 \lor x_{23} \\
x_5 \lor \overline{x}_7 \lor \overline{x}_9 \\
\vdots
\]

**Max-Cut**

\[
x_1 \neq x_2 \\
x_2 \neq x_5 \\
x_3 \neq x_4 \\
\vdots
\]
Max-k-CSP

- $n$ Boolean variables, $m$ constraints (each on $k$ variables)
- Satisfy as many as possible.

Max-3-SAT

$x_1 \lor x_22 \lor \overline{x}_{19}$
$x_3 \lor \overline{x}_9 \lor x_{23}$
$x_5 \lor \overline{x}_7 \lor \overline{x}_9$

\[ \vdots \]

Max-Cut

$x_1 \not= x_2$
$x_2 \not= x_5$
$x_3 \not= x_4$

\[ \vdots \]

One of the most fundamental classes of optimization problems.
Max-3-XOR: Linear equations modulo 2 (in $\pm 1$ variables)
Max-k-CSP

Max-3-XOR: Linear equations modulo 2 (in \( \pm 1 \) variables)

\[
\begin{align*}
x_5 \cdot x_9 \cdot x_{16} &= 1 \\
x_6 \cdot x_{12} \cdot x_{22} &= -1 \\
x_7 \cdot x_8 \cdot x_{15} &= -1 \\
\vdots
\end{align*}
\]
Max-k-CSP

Max-3-XOR: Linear equations modulo 2 (in ±1 variables)

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\[ x_5 \cdot x_9 \cdot x_{16} = 1 \]
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Max-k-CSP($f$): Given predicate $f : \{-1, 1\}^k \rightarrow \{0, 1\}$. Each constraint is $f$ applied to some $k$ (possibly negated) variables.
Max-k-CSP

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\[
C_i \equiv f \left( x_{i_1} \cdot b_1^{(i)}, \ldots, x_{i_k} \cdot b_k^{(i)} \right)
\]
Approximating Max-k-CSP

Relax the problem of finding maximum fraction of constraints satisfiable.
Approximating Max-k-CSP

Relax the problem of finding **maximum fraction** of constraints satisfiable.

$$\leq \theta \quad \text{and} \quad > \gamma \cdot \theta$$

$$\gamma \geq 1$$
Approximating Max-k-CSP

Relax the problem of finding \textit{maximum fraction} of constraints satisfiable.

\[
\leq \theta \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad > \gamma \cdot \theta
\]

\((\gamma \geq 1)\)

- Can solve for all \(\theta\) \(\implies\) Can approximate within factor \(\gamma\).
Approximating Max-k-CSP

Relax the problem of finding maximum fraction of constraints satisfiable.

- Can solve for all $\theta$ $\iff$ Can approximate within factor $\gamma$.
- Hard to solve for some $\theta$ $\iff$ Hard to approximate within factor $\gamma$.
Approximation Resistance

- Let $\rho(f) = \mathbb{E}_x[f(x)]$ be the fraction of constraints satisfied by a random assignment.
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- $\rho(3\text{-SAT}) = 7/8$, $\rho(3\text{-XOR}) = 1/2$
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- $f$ is **approximation resistant** if it is (NP/UG-) hard to distinguish

\[
\leq \rho(f) + \epsilon \quad \geq 1 - \epsilon
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- Captures the notion of when is it hard to do better than a random assignment.
(Sufficient) Conditions for Approximation Resistance

- [Håstad 01]: k-SAT and k-XOR are approximation resistant.
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- [AK 13*]: Characterization when \( f \) is even and instance is required to be \( k \)-partite.
Strong Approximation Resistance

- $f$ is approximation resistant if it is (NP/UG-) hard to distinguish

$$\leq \rho(f) + \epsilon \quad \text{and} \quad \geq 1 - \epsilon$$

- When is it hard to do anything different from a random assignment.
- Equivalent to approximation resistance for odd predicates. Almost all previous results prove strong approximation resistance.
Strong Approximation Resistance

- f is approximation resistant if it is (NP/UG-) hard to distinguish
  \[ \leq \rho(f) + \epsilon \quad \geq 1 - \epsilon \]

- f is strongly approximation resistant if it is (NP/UG-) hard to distinguish
  \[ [\rho(f) - \epsilon, \rho(f) + \epsilon] \quad \geq 1 - \epsilon \]
Strong Approximation Resistance

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Strong Approximation Resistance

- f is **approximation resistant** if it is (NP/UG-) hard to distinguish

\[
\leq \rho(f) + \epsilon \quad \geq 1 - \epsilon
\]

- f is **strongly approximation resistant** if it is (NP/UG-) hard to distinguish

\[
[\rho(f) - \epsilon, \rho(f) + \epsilon] \quad \geq 1 - \epsilon
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A partial characterization by [Rag 08] and [RS 09]

- [Rag 08\*]: \( f \) is approximation resistant iff \( \forall \epsilon > 0 \) there exists a \( 1 - \epsilon \) vs. \( \rho(f) + \epsilon \) integrality gap instance for a certain SDP.

- Above argument also works for strong approximation resistance.

Gives a recursively enumerable condition.

- But what properties of \( f \) give rise to gap instances?

- Is it just properties of \( f \) or is the topology of the instance also important?

(Hint: Just \( f \))
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- Above argument also works for strong approximation resistance. Gives a recursively enumerable condition.

- But **what properties of $f$** give rise to gap instances?

- Is it just properties of $f$ or is the topology of the instance also important? (Hint: Just $f$)
The Austrin-Mossel condition in a new language

- For a distribution $\mu$ on $\{-1, 1\}^k$, let $\zeta(\mu) \in \mathbb{R}^{k+\binom{k}{2}}$ denote the vector of first and second moments

$$
\zeta_i = \mathbb{E}_{x \sim \mu}[x_i] \quad \zeta_{ij} = \mathbb{E}_{x \sim \mu}[x_i \cdot x_j]
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- Let $C(f)$ be the convex polytope

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C(f) = \left\{ \zeta(\mu) \mid \mu \text{ is supported on } f^{-1}(1) \right\}.
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- [AM 09*]: $f$ is (strongly) approximation resistant if $0 \in C(f)$.

- Our condition is in terms of existence of a measure $\Lambda$ on $C(f)$ with certain symmetry properties.
Transformations of a measure $\Lambda$ on $C(f)$

- Each $\zeta \in C(f)$ can be transformed by:

  - Permuting the underlying $k$ variables by a permutation $\pi$: $\zeta_{\pi i} = \zeta_{\pi(\pi^{-1}(i))}$
  
  - Multiplying each variable $x_i$ by a sign $b_i \in \{-1, 1\}$: $\zeta_{b_i} = b_i \cdot \zeta_i$
  
  - Projecting $\zeta$ to coordinates corresponding to a subset $S \subseteq [k]$. For $S \subseteq [k]$, $\pi : S \to S$, $b \in \{-1, 1\}$, let $\Lambda_{S, \pi, b}$ denote the measure obtained by transforming each point in support of $\Lambda$ as above.

- If $\Lambda$ is supported only on 0, then so is each $\Lambda_{S, \pi, b}$. If $\Lambda$ is supported only on (say) $(1, \ldots, 1)$ then $\Lambda_{[k], \text{id}, b}$ is supported only on the point $(b_1, \ldots, b_k, b_1 \cdot b_2, \ldots, b_{k-1} \cdot b_k)$.
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Transformations of a measure \( \Lambda \) on \( C(f) \)

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- If $\Lambda$ is supported only on 0, then so is each $\Lambda_{S,\pi,b}$. If $\Lambda$ is supported only on (say) $(1, \ldots, 1)$ then $\Lambda_{[k],\text{id},b}$ is supported only on the point $(b_1, \ldots, b_k, b_1 \cdot b_2, \ldots, b_{k-1} \cdot b_k)$.
Recall that $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ can be written as

$$f(x) = \sum_{S \subseteq [k]} \hat{f}(S) \cdot \prod_{i \in S} x_i = \rho(f) + \sum_{t=1}^{k} \sum_{|S|=t} \hat{f}(S) \cdot \prod_{i \in S} x_i$$
Our Characterization

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- [KTW 13*]: $f$ is strongly approximation resistant if and only if there exists a measure $\Lambda$ on $C(f)$ such that for all $t = 1, \ldots, k$

$$\sum_{|S|=t} \sum_{\pi:S \rightarrow S} \sum_{b \in \{-1, 1\}^S} \hat{f}(S) \cdot \left( \prod_{i \in S} b_i \right) \cdot \Lambda_{S,\pi,b} \equiv 0$$
Our Characterization

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$$\sum_{|S|=t} \sum_{\pi:S \rightarrow S} \sum_{b \in \{-1,1\}^S} \hat{f}(S) \cdot \left(\prod_{i \in S} b_i\right) \cdot \Lambda_{S,\pi,b} \equiv 0$$

- If $|S| = t$, then $\Lambda_{S,\pi,b}$ is a measure on $\mathbb{R}^{t+\binom{2}{t}}$. For each $t$, above expression is a linear combination of such measures.
Proof Structure

- No good \( \Lambda \) exists
- Good \( \Lambda \) exists
Proof Structure

No good
Λ exists

Good
Λ exists

Standard PCP ideas

Hardness
Proof Structure

No good \( \Lambda \) exists

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Standard PCP ideas

Hardness

Algorithm

zero-sum game

value > 0

Algorithm

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value = 0

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No good $\Lambda$ exists

Good $\Lambda$ exists

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value $> 0$

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Algo

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\[ \text{Algorithm} \]

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The (infinite) two-player game

- Similar game also used by O’Donnell and Wu for Max-Cut.
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- Hardness player tries to design an integrality-gap instance. Each constraint has local distribution $\mu$ with moments given by $\zeta(\mu)$. Plays measure $\Lambda$ on $C(f)$ (corresponds to instance).

- Algorithm player tries to round by first projecting to random $d$-dimensional Gaussian. Plays rounding strategy $\psi: R^d \to \{-1, 1\}$.

- Value $= |\rho(f) - \text{Expected fraction of constraints satisfied by } \psi| > 0$ implies (a distribution over) rounding strategies which show that predicate is not strongly approximation resistant. (since every instance corresponds to a $\Lambda$)
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- Value = \(|\rho(f) - \text{Expected fraction of constraints satisfied by } \psi|\)

- Value > 0 implies (a distribution over) rounding strategies which show that predicate is not strongly approximation resistant. (since every instance corresponds to a \( \Lambda \))
Value of the game

- A random constraint in the instance corresponds to $\zeta \sim \Lambda$. 
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Obtaining conditions on $\Lambda$ when value $= 0$

- Value $= \left| \mathbb{E}_{\zeta \sim \Lambda} \mathbb{E}_{y_1, \ldots, y_k \sim N(\zeta)} \left[ \sum_{S \neq \emptyset} \hat{f}(S) \cdot \prod_{i \in S} \psi(y_i) \right] \right|$. 

- There exists (distribution over) $\Lambda$ which gives value 0 for all $\psi$.
- Value can be viewed as a polynomial in the infinitely many variables $\psi(y)$ for $y \in \mathbb{R}^d$ which is zero for all assignments $\psi$.
- All coefficients must be 0. Coefficients are linear combinations of integrals of $\Lambda S, \pi, b$ w.r.t. some Gaussian densities.
- Need to conclude integrals are zero only if the corresponding linear combinations are 0. Degree $t$ coefficients give condition at level $t$.
- Bulk of the work in analyzing sequence of finite games and coefficients of corresponding polynomials.
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Concluding Remarks

- We also characterize
  - Approximation resistance for odd predicates (including threshold functions passing through origin).
  - Approximation resistance for \(k\)-partite instances (all predicates).
  - Sherali-Adams LP gaps for \(\omega(1)\) levels (all predicates).
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- **Problem**: Strong Approximation Resistance vs. Approximation Resistance.
Thank You

Questions?