Anderson Localization at the 1D edge of a 2D topological insulator

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2D Topological Insulator

Kane and Mele (2005); Bernevig, T. L. Hughes, and S. C. Zhang (2006)

\[
\hat{H}(k_x) = v_F k_x \cdot \sigma_z
\]

Hamiltonian of the chiral states at the helical edge

Note: need strong spin-orbital interaction
Rashba term:

\[ H_{S-O} \propto \vec{E} \bullet [\vec{\sigma} \times \vec{k}] \rightarrow \pm v_F k \]
2D Topological Insulator

Kane and Mele (2005);

Hamiltonian of the chiral edge states

\[ \hat{H}(k_x) = v_F k_x \cdot \sigma_z \]

momentum
Fermi velocity
Z-component of the spin

Chiral edge states: Left and Right movers
Chiral edge states: Left and Right movers

Backscattering would mix the chiral states and thus destroy chirality.

One needs spin-flip for the backscattering.

Kamers degeneracy: one needs to violate the time-reversal symmetry to mix left and right movers.
Basic properties of a generic 2D Topological Insulator:

(strong spin-orbit coupling)

2D bulk = insulator: electron spectrum is gapped, levels of impurities are localized

Edge Modes are Helical

Statement:

Time Reversal Symmetry protects Helical Edge Modes from Backscattering and thus from Anderson Localization
Quantum Hall Effect: spatially separated edge states - nowhere to scatter

Topological Insulator: the back moving state is nearby but the back-scattering can not happen without a spin flip
Topological Insulator: metallic edge

Topological Insulator: *Anderson localization at the edge*
Time Reversal Symmetry protects Helical Edge Modes from Backscattering and Anderson Localization

Conductance of an ideal 1D helical edge should be:

\[ G_{\text{ideal}} = \frac{2e^2}{h} \]

In reality, conductance of only few short samples is close to \( G_{\text{ideal}} \), while for longer samples \( G \ll G_{\text{ideal}} \).
Experimental Observations

- Large samples show large resistance at the gap.
- Small samples (~1X1μm) show quantized conductance at the gap, indicating transport by edge states.
- $g$: 20-50

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<thead>
<tr>
<th></th>
<th>$d$ (nm)</th>
<th>$L \times W$ (μm²)</th>
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</thead>
<tbody>
<tr>
<td>I</td>
<td>5.5</td>
<td>20.0×13.3</td>
</tr>
<tr>
<td>II</td>
<td>7.3</td>
<td>20.0×13.3</td>
</tr>
<tr>
<td>III</td>
<td>7.3</td>
<td>1.0×1.0</td>
</tr>
<tr>
<td>IV</td>
<td>7.3</td>
<td>1.0×0.5</td>
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$l_{inelastic} \sim 1\mu m$

[Molenkamp’s group]
Yacoby’s group; private communication
Questions:

- How universal is the protection?
- Can the “topological protection” be softened?
- Can helical edge electrons be localized?
Time Reversal Symmetry can be trivially broken by an External Magnetic Field.

Spatially homogeneous field – no effect!
Modulated \( \sim \cos(2k_F x) \) field \( \rightarrow \) energy gap
Homogeneous field + potential disorder = backscattering!

Q: What about intrinsic sources of the Time Reversal Symmetry Violation?
Localized Spins in the presence of itinerant electrons = Kondo Spins

**Origin:**
1. Chemistry: dangling bonds, etc.
2. Localized energy levels close to the edge

Onsite Hubbard repulsion – Anderson Model
4 states of each localized level. Different energies

Chemical potential $\mu$

$E_c$ – energy of the repulsion

$E_c > \mu \Rightarrow \begin{cases} E_0 > E_1 \\ E_2 > E_1 \end{cases}$

Anderson Model $\Rightarrow$ Kondo Model
Magnetic (spin) impurity near the helical edge

Electron-spin interaction:

\[ U(1): \quad J_z \sigma^z S^z + J_\parallel (\sigma^x S^x + \sigma^y S^y) = \]
\[ = J_z \sigma^z S^z + \frac{J_\parallel}{2} (\sigma^+ S^- + \sigma^- S^+) \]

No influence on the \( T = 0 \) dc charge transport

Reasons:
$U(1)$-symmetric (xy-isotropic) electron-spin interaction has no influence on zero-temperature dc charge transport.

1. **Kinematic reason** (Tanaka, Furusaki, Matveev (2011))

Spin down impurity can back-scatter a right-moving electron.

However, subsequent backscattering of right-moving electrons is impossible until some left-moving electron reverse the impurity spin!
$U(1)$ - symmetric (xy-isotropic) electron-spin interaction has no influence on zero-temperature dc charge transport

Second reason: Kondo effect - screening of the impurity spin

Recovery of the Time Reversal Symmetry

\[ \frac{e^{i\varphi}}{\sqrt{2}} |\uparrow\rangle + \frac{e^{-i\varphi}}{\sqrt{2}} |\downarrow\rangle \]
Single spin: T-invariance always survives due to the Kondo effect

Finite density of spins: T-invariance can be violated spontaneously (Kondo — RKKY) but the backscattering would not appear as long as the z-component of the total spin is conserved, i.e. as long as the system remains $U(1)$-symmetric, i.e. invariant under rotations in spin space around z-axis.

Q: What if there is a small but finite density of localized and anisotropic spins?
Finite temperature effects of the localized electron and nuclear spins without spontaneous breaking of the time reversal symmetry:


B. Braunecker, P. Simon, and D. Loss, Phys. Rev. B 80, 165119 (2009)

Q: What if there is a small but finite density of localized and anisotropic spins?

Kondo \quad \rightarrow \quad RKKY \quad \rightarrow \quad Spin \玻璃

density of spins

electron-spin coupling constant

T-invariant

Broken T-invariance

Spontaneous breaking of the time-reversal symmetry
<table>
<thead>
<tr>
<th>No magnetic anisotropy $U(1)$</th>
<th>Magnetic anisotropy $S_z^{tot}$ is not conserved $U(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_z^{tot}$ is conserved</td>
<td></td>
</tr>
<tr>
<td><strong>No disorder</strong> (regular spin chain)</td>
<td><strong>Perfect 1D metal</strong> $\sigma = 2e^2/h$</td>
</tr>
<tr>
<td><strong>Band Insulator:</strong></td>
<td>Charged excitations are gapped even at the edge</td>
</tr>
<tr>
<td>Disorder</td>
<td><strong>Goldstone mode</strong> perfect 1D metal $\sigma = 2e^2/h$</td>
</tr>
<tr>
<td><strong>Anderson Insulator:</strong></td>
<td>Edge states are localized</td>
</tr>
</tbody>
</table>
Localized spin impurities $\vec{S}_j$, located at $x_j$, $x_{j+1} > x_j$

Linear spin density

$$\rho_s(x) = \sum_j \delta(x - x_j);$$

Averaged spin density:

$$\rho_s \equiv \langle \rho_s(x) \rangle \equiv \frac{1}{a}$$
Hamiltonian:

\[ H = H_e + H_{e-S} \]

Free helical electrons:

\[ H_e = -i v_F \int dx \hat{\Psi}^\dagger(x) \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \hat{\Psi}(x) = \]

\[ = -i v_F \int dx \left[ \hat{\Psi}_R^{\dagger}(x) \partial_x \Psi_R^{(\uparrow)} - \hat{\Psi}_L^{\dagger}(x) \partial_x \Psi_L^{(\downarrow)} \right] \]

Electron-spin interaction: \( \tilde{\sigma} \) - electron spin (\( \tilde{\sigma} = \hat{\Psi}^\dagger \hat{\sigma} \hat{\Psi} \))

\[ U(1): \quad J_z \sigma_z S^z + J_\parallel (\sigma_x S^x + \sigma_y S^y) = J_z \sigma_z S^z + \frac{J_\parallel}{2} (\sigma_+ S^- + \sigma_- S^+) \]

\[ \times U(1): \quad J_z \sigma_z S^z + J_x \sigma_x S^x + J_y \sigma_y S^y = J_z \sigma_z S^z + \frac{J_\parallel}{2} (\sigma^+ S^- + \sigma^- S^+) + \frac{\delta J}{2} (\sigma^+ S^+ + \sigma^- S^-) \]

\[ J_\parallel = \frac{1}{2} (J_x + J_y), \quad \delta J = \frac{1}{2} (J_x - J_y) \]

| \delta J | << J_\parallel

Note: the Hamiltonian is \( T \)-invariant
$U(1)$: Electron-spin interaction

$$H_{e-S} = \sum_j \left[ J_z S^z \left( \Psi_R^{+\uparrow} \Psi_R^{\uparrow} - \Psi_L^{+\downarrow} \Psi_L^{\downarrow} \right) + \frac{J_\parallel}{2} \left( S_j^+ \Psi_L^{+\downarrow} e^{2ik_Fx_j} \Psi_R^{\uparrow} + S_j^- \Psi_R^{+\uparrow} e^{-2ik_Fx_j} \Psi_L^{\downarrow} \right) \right]$$

**Preliminaries:**

Effective spin-spin ("RKKY") interaction

**Features of chirality:**

1. No $S^z_j S^z_k$ interaction
2. Factors $e^{2ik_F(x_j-x_k)}$ instead of $\cos[2k_F(x_j-x_k)]$

**Helical (chiral) electrons**

$$H_{S-S} = -\frac{J^2_\parallel}{8\pi v_F} \sum_{j \neq k} S^+_j S^-_k e^{2ik_F(x_j-x_k)} + h.c.$$ 

**Usual (non-chiral) electrons**

$$H_{S-S} \propto \frac{J^2}{v_F} \sum_{j \neq k} \tilde{S}_j \tilde{S}_k \cos[2k_F(x_j-x_k)]$$

Can be removed by

$$S_j^\pm \leftarrow \tilde{S}_j^\pm = S_j^\pm e^{\mp2ik_Fx_j}$$
Effective spins with ferromagnetic exchange interaction

\[ \tilde{S}_j^\pm = S_j^\pm e^{\mp 2ik_F x_j} \]

Ferromagnet rather than Spin Glass without mean magnetization

Quasiferromagnet?

\[ H_{S-S}^{(eff)} = - \frac{J^2}{8\pi n_F} \sum_{j\neq k} \frac{\tilde{S}_j^+ \tilde{S}_k^- + h.c.}{|x_j - x_k|} \]

Insulating bulk

Localized spins

Left mover

Spin down

\sim a

Right mover

Spin up

Helical edge
Coherent representation of the spins:

$\vec{S}_j$ is parametrized by $\alpha_j$ and $n_{j,z}$

$S_{j}^{z} \rightarrow \frac{1}{2} n_{z,j}$; $S_{j}^{\pm} \rightarrow \frac{1}{2} \sqrt{1 - n_{z,j}^{2}} e^{\pm i\alpha_j}$

$\vec{S}_j \rightarrow \frac{1}{2} \vec{n}_j$; $|\vec{n}_j| = 1$
Effective spin-spin interaction (for rotated spins) in the coherent spin representation:

\[
H_{S-S}^{(\text{eff})} = -\frac{J^{\parallel} S^2}{4\pi v_F} \sum_{j \neq k} \frac{\sqrt{(1 - n_{z,j}^2)(1 - n_{z,k}^2)} \cos(\alpha_j - \alpha_k)}{|x_j - x_k|}
\]

“Classical” ground state:

\[
n_{z,j} = 0, \quad \alpha_j = \text{Const}
\]

Task: to account for quantum fluctuations

Slow and long-range fluctuations

continual description

all spins are in xy plane
ferromagnetic order

\[
n_{z,j} \leftrightarrow n_z(x)
\]

\[
\alpha_j \leftrightarrow \alpha(x)
\]

\[
\sum_j \rightarrow \int dx \rho_s(x)...
\]
Partition function in functional integral representation

\[ Z = \int D\hat{\Psi}^\dagger D\hat{\Psi} Dn_z(x) D\alpha(x) \exp[-S(\hat{\Psi}^\dagger, \hat{\Psi}; n_z, \alpha)] \]

Flat (!) Integration measure:

\[ D\tilde{S} = \prod_j dn_{z,j}(\tau) d\alpha_j(\tau) \to Dn_z(\tau, x) D\alpha(\tau, x) \]

Electron operator:

\[ \hat{\Psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{R\uparrow} e^{ik_Fx} \\ \Psi_{L\downarrow} e^{-ik_Fx} \end{pmatrix} \]

Matsubara action, \( \tau \) is imaginary time

\[ \mathcal{S}(\hat{\Psi}^\dagger, \hat{\Psi}; \tilde{S}) = \int dx d\tau \hat{\Psi}^\dagger \begin{pmatrix} \partial_\tau & 0 \\ 0 & \partial_\tau \end{pmatrix} \hat{\Psi} + \int dx d\tau H_{e-S}(\hat{\Psi}^\dagger, \hat{\Psi}; \tilde{S}) + \mathcal{S}_S \]

Matsubara action, \( \tau \) is imaginary time

\[ \mathcal{S}_S = -i S \int \rho_s(x) dx \int_0^\beta d\tau [1 - n_z(\tau, x)] \partial_\tau \alpha(\tau, x) \]
Matsubara Action (\(U(1)\)-symmetric)

\[
S = \int \rho_S(x) dx \int_0^\beta d\tau \left( L_{e-S} + L_S \right)
\]

\[
L_{e-S} = \hat{\Psi}^\dagger \begin{pmatrix}
\partial_+ & \Delta \sqrt{1 - n_z^2} e^{i\alpha} \\
\Delta \sqrt{1 - n_z^2} e^{i\alpha} & \partial_-
\end{pmatrix} \hat{\Psi}
\]

\[
L_S = -i S \left[ 1 - n_z(\tau, x) \right] \partial_\tau \alpha(\tau, x)
\]

Note: if \(\alpha(\tau, x) = const\) then the electrons are gapped!

Gauge transformation to account for variations of \(\alpha(x, \tau)\)

\[
\hat{\Psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\Psi_{R \uparrow} e^{i k_F x} \\
\Psi_{L \downarrow} e^{-i k_F x}
\end{pmatrix}
\]

\[
\partial_\pm = \partial_\tau \pm iv_F \partial_x
\]

\[
\Delta = \frac{S \rho_S J_{||}}{2}
\]

\[
\Psi_{R \uparrow} = e^{-\frac{i\alpha}{2}} \Psi_{R \uparrow}, \quad \Psi_{L \downarrow} = e^{\frac{i\alpha}{2}} \Psi_{L \downarrow}
\]

\[
\Psi^\dagger_{R \uparrow} = e^{\frac{i\alpha}{2}} \overline{\Psi}_{R \uparrow}, \quad \Psi^\dagger_{L \downarrow} = e^{-\frac{i\alpha}{2}} \overline{\Psi}_{L \downarrow}
\]

\[
L_{e-S} = \hat{\Psi}^\dagger \begin{pmatrix}
\partial_+ - i \partial_+ \alpha/2 & \Delta \sqrt{1 - n_z^2} \\
\Delta \sqrt{1 - n_z^2} & \partial_- + \frac{i}{2} i \partial_+ \alpha/2
\end{pmatrix} \hat{\Psi}
\]
Matsubara Action ($U(1)$-symmetric)

\[ S = \int \rho_S(x) dx \int_0^\beta d\tau \left( L_{e-s} + L_s \right) \]

\[ L_{e-s} = \hat{\Psi}^\dagger \begin{pmatrix} \partial_+ + i\partial_+ \alpha / 2 & \Delta \sqrt{1-n_z^2} \\ \Delta \sqrt{1-n_z^2} & \partial_- + i2i \partial_+ \alpha / 2 \end{pmatrix} \hat{\Psi} \]

\[ L_s = -iS[1-n_z(\tau, x)] \partial_\tau \alpha(\tau, x) \]

\[ \hat{\Psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{R\uparrow} e^{ik_F x} \\ \Psi_{L\downarrow} e^{-ik_F x} \end{pmatrix} \]

\[ \partial_\pm = \partial_\tau \mp iv_F \partial_x \]

\[ \Delta = \frac{S \rho_s J_{\parallel}}{2} \]

\[ Z = \int D\hat{\Psi}^\dagger D\hat{\Psi} Dn_z(x) D\alpha(x) e^{-S} \]

**Need to**

1. Integrate over $n_z$
2. Integrate over $\hat{\Psi}(x)$
3. Take into account the chiral anomaly

\[ S[\alpha] = \int \rho_S(x) dx \int_0^\beta d\tau \frac{1}{2\pi K} \left\{ \frac{1}{u} (\partial_\tau \alpha)^2 + u (\partial_x \alpha)^2 \right\} \]

\[ K = \frac{u}{v_F} \]

\[ u = \frac{J_{\parallel}}{2\pi} \sqrt{\ln \left( \frac{E_B}{\Delta} \right)} \]
Matsubara Action ($U(1)$-symmetric)

$$S[\alpha] = \int \rho_s(x) \, dx \int_0^\beta d\tau \frac{1}{2\pi K} \left\{ \frac{1}{u} (\partial_\tau \alpha)^2 + u (\partial_x \alpha)^2 \right\}$$

**Luttinger Liquid!**

$$u = \frac{J_\parallel}{2\pi} \sqrt{\ln \left( \frac{E_B}{\Delta} \right)} \ll v_F$$

$$K = \frac{u}{v_F} = \left( \frac{J_\parallel}{2\pi v_F} \right) \sqrt{\ln \left( \frac{E_B}{\Delta} \right)} \ll 1$$

**Valid provided that**

$$J_\parallel \ll v_F$$

**Effective velocity**

**Luttinger parameter**

Charge density

$$\rho_{el} = -\frac{1}{2\pi} \partial_x \alpha$$

Free bosonic field Goldstone mode

$$\alpha(\tau, x)$$

**Ideal Conductance**
Free chiral edge electrons
Electron-spin exchange
$U(1)$ - symmetry

Luttinger Liquid
with small velocity
and
small Luttinger parameter

Ideal Metallic Conductance
even in the presence of a potential disorder
No disorder: \( u = \text{const}; \quad \Delta = \text{const} \)

**Density-density correlation function**

\[
\langle \rho_{el}(-q,-\omega)\rho_{el}(q,\omega) \rangle = \frac{v_F q^2}{2\pi} \left[ \frac{1}{6\Delta^2} + \frac{u^2}{v_F^2} \frac{1}{\omega^2 + u^2 q^2} \right]
\]

**Current-current correlation function**

\[
\langle j_{el}(-q,-\omega) j_{el}(q,\omega) \rangle_{q,\omega \to 0} = \frac{e^2 \omega^2}{q^2} \langle \rho_{el}(-q,-\omega)\rho_{el}(q,\omega) \rangle \sim \frac{e^2}{2\pi} \frac{u^2}{v_F^2} \frac{\omega^2}{\omega^2 + u^2 q^2}
\]

**Conductivity is ballistic**

\[
\sigma(\Omega) = \frac{\langle j_{el}(-q,-\omega) j_{el}(q,\omega) \rangle_{q=0,i\omega \to \Omega}}{i\Omega} \propto \frac{1}{\Omega}
\]

**Potential Disorder:**

\[
u = \text{const} + \delta u(x); \quad \Delta = \text{const} + \delta u(x)
\]

**Scattering rate:**

\[
\Gamma(\omega) \propto \frac{\omega^2}{\Delta} \quad \Rightarrow \quad \text{No effect on dc conductivity!}
\]

\( U(1) \) symmetry is not violated
**Matsubara Action (broken $U(1)$-symmetry)**

**Electron-spin interaction:**

$$\frac{J ||}{2}(\sigma_j^+ S_j^- + \sigma_j^- S_j^+) + \frac{\delta J_j}{2}(\sigma_j^+ S_j^+ + \sigma_j^- S_j^-)$$

**Anisotropy of $j$-th spin**

$$\varepsilon_j \propto \frac{\delta J_j}{J ||} e^{4i k_F x_j}$$

**If $\varepsilon_j$ is independent of $j$, then the spectrum is gapfull**

**More likely $\varepsilon_j$ is random**

$$\langle \varepsilon_k^* \varepsilon_j \rangle = d \delta_{jk}$$

**Measure of the $U(1)$ breaking disorder**

**Continuous limit**

$$\varepsilon_j \leftarrow \varepsilon(x)$$

$$\langle \varepsilon(x) \varepsilon^*(x') \rangle = \rho_s d \delta(x - x')$$
Matsubara Action (broken $U(1)$-symmetry)

**Electron-spin interaction:**
\[
\frac{J_\parallel}{2} (\sigma_j^+ S_j^- + \sigma_j^- S_j^+) + \frac{\delta J_j}{2} (\sigma_j^+ S_j^+ + \sigma_j^- S_j^-)
\]

\[
\mathcal{E}_j \propto \frac{\delta J_j}{J_\parallel} e^{4ik_F x_j}
\]

**Modified electron-spin action:**
\[
S_{e-S}[\Psi, \bar{S}] = \int d\tau dx \hat{\Psi}^\dagger \begin{bmatrix}
\partial_+, & \Delta \sqrt{1-n_z^2} [e^{-i\alpha} + \mathcal{E}(x)e^{i\alpha}]

\Delta \sqrt{1-n_z^2} [e^{i\alpha} + \mathcal{E}^*(x)e^{-i\alpha}], & \partial_-
\end{bmatrix} \hat{\Psi}
\]

**Modified boson action:**
\[
S[\alpha] \rightarrow \int d\tau dx \left[ \frac{v_F}{8\pi u^2} \left[ (\partial_\tau \alpha)^2 + u^2 (\partial_x \alpha)^2 \right] + \text{Re} \left[ \mathcal{E}(x)e^{2i\alpha(x,\tau)} \right] \right]
\]
Matsubara Action (broken $U(1)$-symmetry)

\[
S[\alpha] \rightarrow \int d\tau dx \left[ \frac{v_F}{8\pi u^2} \left( (\partial_\tau \alpha)^2 + u^2 (\partial_x \alpha)^2 \right) + \text{Re} \left[ \epsilon(x) e^{2i\alpha(x,\tau)} \right] \right]
\]

\[
\langle \epsilon^*(x) \epsilon(x') \rangle = \rho_s d \delta(x-x')
\]

**Mapping on the problem of the pinning of one-dimensional charge-density wave by potential disorder.** (Giamarchi & Schulz, 1988)

**Localization length**

\[
L_{loc} = a \left( \frac{v_F}{J_\parallel} \right)^{2-2K} \left( \frac{1}{d \ln \left( \frac{E_B}{\Delta} \right)} \right)^{\frac{1}{3-2K}}
\]

\[
u = \frac{J_\parallel}{2\pi} \left( \ln \frac{E_B}{\Delta} \right)^{1/2} \ll v_F, \quad \Delta = \frac{\rho_s J_\parallel}{4}
\]

\[
K = \frac{u}{v_F} = \frac{J_\parallel}{2\pi v_F} \left( \ln \frac{E_B}{\Delta} \right)^{1/2} \ll 1
\]
Conclusions

Random interactions of helical edge electrons with a closely located spin (Kondo) impurities lead to Anderson localization of electrons if the total z-component of spin is not conserved (i.e. $U(1)$ symmetry is broken)

Physical interpretation:
1. In the presence of $U(1)$ symmetry phases of spin impurities rotate which restores effectively the time reversal symmetry and prevents localization;
2. If $U(1)$ symmetry is randomly broken, the phase of spins is pinned which means a spontaneous breaking of T-invariance, i.e. no protection against Anderson localization
Open Questions (under way):

Electron-electron interaction effects,
Nonlinear response, ....
Q: Is the topological insulator a distinct new state of matter or the main conclusion is that the localization length at the edge can be much bigger than in the bulk?
To take home:

Albeit electrons at 1d helical edge of a 2D topological insulator are more protected from an influence of random imperfections, they are still subjected to Anderson localization similar to the usual 1D conductors.

Thank you!
Our case:

\[ K = \frac{u}{v_F} \ll 1 \]
Comparison with BKT and justification of the approach

Boson field action of the considered 1D quantum system:

\[ S[\alpha] = \frac{v_F}{8\pi u^2} \int d\tau dx [(\partial_\tau \alpha)^2 + u^2 (\partial_x \alpha)^2] = \frac{v_F}{8\pi u} \int dx^0 dx^1 (\nabla' \alpha)^2 \]

\[ (x^0 = u\tau, x^1 = x) \]

Action of classical 2D X-Y model:

\[ S[\alpha] = \frac{\beta J}{2} \int d^2 x (\nabla \alpha)^2 \quad (\beta = \frac{1}{T}) \]

BKT phase transition:

\[ \frac{\beta J}{2} > \frac{\beta_{cr} J}{2} = \frac{1}{\pi} \quad - \text{ordered low-temperature phase (no real vortices)} \]

\[ \frac{\beta J}{2} < \frac{\beta_{cr} J}{2} = \frac{1}{\pi} \quad - \text{disordered high-temperature phase (vortices)} \]

Our case: \( \frac{v_F}{8\pi u} \gg \frac{1}{\pi} \) - no vortices!
2D Topological Insulators

Volkov & Pankratov (1985)

\[ d_c \approx 6.35 \text{nm} \]